Presheaves, Sheaves and their Topoi in Quantum Gravity and Quantum Logic[*]

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Abstract

A brief synopsis of recent conceptions and results, the current status and future outlook of our research program of applying sheaf and topos-theoretic ideas to quantum gravity and quantum logic is presented.

"*Logics come from dynamics*"
(D. R. Finkelstein: '*Quantum Relativity*' [29])

1 Introduction: two questions motivating our quest

The following two questions, one physical the other more mathematical, motivate essentially our general research project of applying sheaf and topos-theoretic concepts, techniques and results to quantum gravity and quantum logic:

- *Is there a fundamental connection between the quantum logical structure of the world and its dynamically variable microcausal or chronological structure at Planck scales as the latter is supposed to be determined by the until now persistently elusive quantum theory of gravity?*

and related to it:

- *How can one localize noncommutatively?*

Concerning the first question, we intuit that a sound theoretical scheme for quantum gravity should be intimately related to the logical structure of the world at quantum scales: in a strong sense, quantum causality should be unified at the dynamical level with quantum logic in the light of quantum gravity. In turn, this conjecture essentially implies our main suspicion that in the quantum spacetime deep even quantum logic should be regarded as a quantum 'observable' entity that is subject to dynamical changes—a *dynamical physical logic* analogous to the dynamical physical spacetime geometry of the classical theory of gravity (*ie*, general relativity) [25, 26]. That *logos is somehow related to chronos at a fundamental level*[1] has become the central theme in our quantum gravity research program over the last few years.

Our subsequent decision to implement mathematically this theme by using sheaf and topos-theoretic concepts, techniques and results is based on the by now widely established fact, at least among categorists and related 'toposophers', that *the theory of presheaves, sheaves and their topoi fuses geometry with logic at a basic level* [40, 31, 42]. It only appeared natural to us that if geometry could be somehow identified with 'spacetime geometry' in particular, while logic with 'quantum logic', then the long sought after unification of relativity with quantum mechanics could be possibly achieved by sheaf and topos-theoretic means. After all, the methods of sheaf and topos theory are of an essentially algebraic nature [42, 43, 44], and lately there has been a strong tendency among mathematical physicists to tackle the problem of quantum gravity entirely by categorico-algebraic means [17, 45, 46, 47, 48].

Concerning the second motivating question above which, as we will argue subsequently, is closely related to the first, our quest focuses on a possible formulation of a *noncommutative topology* and its associated *noncommutative sheaf theory* that can can be applied to the problem of the quantum structure and dynamics

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[1]In fact, the opening quotation from [29] suggests that *logic derives from dynamics*!

of spacetime. Our original motivation for looking into the possibility of a noncommutative or, ultimately, 'quantum' topology for quantum gravity rested heavily on our desire to abandon the geometric spacetime continuum on which the mathematics of general relativity (*ie*, the standard differential geometry) essentially rests for some structure of a more finitistic, algebraic and, hopefully, dynamical character [61, 47, 62, 48, 58]. This is the subject of the next section and we will use it as the *raison d'être* of our endeavor to apply sheaf and topos-theoretic ideas to quantum gravity.

Thus, the short report below commences with various physical and mathematical evidence that we have collected in the past couple of years against the classical topological (*ie*, $\mathcal{C}^0$-continuous) and differential (*ie*, $\mathcal{C}^\infty$-smooth) manifold model of spacetime by essentially basing ourselves on quantum theory's principles of finiteness or discreteness, superposition and, as a result, algebraic noncommutativity, as well as on relativity's central principle of local causality commonly known as 'locality'. Thus, we will see how effectively sheaf and topos-theoretic ideas may be used to formulate a *locally finite, causal and quantal version of* (at least the kinematics of) *discrete Lorentzian quantum gravity* [61, 56, 57, 47, 62, 48, 58, 60], for it has been convincingly argued that capturing the 'proper' kinematical structure constitutes the first decisive step towards arriving at the notoriously elusive quantum dynamics for spacetime and gravity [73, 62].

2   The Past: manifold reasons against the spacetime manifold

A noncommutative geometry [16] has already been proposed, significantly worked out and diversely applied to the problem of the quantum structure and dynamics of spacetime (*ie*, quantum gravity). However, it seems theoretically rather *ad hoc*, lame and short sighted to think of a higher level structure such as the geometry of spacetime as being subject to some sort of quantization and as participating into, in principle measurable, dynamical variations[2] thus be soundly modelled by noncommutative mathematics, while a more basic structure such as the spacetime topology to be treated essentially as a fixed classical entity, hence be modelled after a non-varying locally Euclidean manifold equipped with algebras of commutative coordinates labelling its point events [61, 47, 58, 62]. Related to this, and from a rather general and technical perspective, while a commutative sheaf theory has been rather quickly developed, well understood and widely applied to both mathematics and physics [33, 24, 69, 43, 44, 45, 46], a noncommutative one (and the topology related to it) has been rather slow in coming and certainly not unanimously agreed on how to be applied to quantum spacetime research [69, 77, 79, 50, 51, 7, 8, 6, 47, 58][3].

At the same time, and from a physical point of view, the unreasonableness and unphysicality of the locally Euclidean topological ($\mathcal{C}^0$) and differential ($\mathcal{C}^\infty$) manifold model M for spacetime is especially pronounced when one considers:

- (a) **Pointedness of events:** M's pathological nature in the guise of singularities that plague general relativity—the classical theory of gravity—which are mainly due to the geometric point-like character of the events that constitute it, as well as due to the algebras of $\mathcal{C}^\infty$-smooth functions employed to coordinatize these point events [48] (and also due to (b) next).

- (b) **Continuous infinity of events:** M's problematic nature due to the fact that one can in principle pack an uncountable infinity of the aforementioned point events in a finite spacetime volume resulting in the non-renormalizable infinities that impede any serious attempt at uniting quantum mechanics with general relativity (at least at the 'calculational' level)[4].

- (c) **Non-dynamical and non-quantal topology:** Its non-variable and non-quantal nature when one expects that at Planck scales not only the spacetime metric, but also that the spacetime topology partakes into quantum phenomena [80], that is to say, it is a dynamically variable entity whose connections engage into coherent quantum superpositions. We may distill this by saying that the manifold topology is, quantally speaking, an *unobservable* entity not manifesting quantum dynamical fluctuations or interference between its defining connections [62, 58]—a rigid substance, once and forever fixed by the theorist, that is not part of the dynamical flux of Nature at microscopic scales.

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[2]That is, in general relativity at least, the gravitational field, which is represented by the spacetime metric $g_{\mu\nu}$, is treated as an *observable*; in fact, the sole spacetime observable.

[3]That is to say, not all mathematicians and mathematical physicists agree on what ought to qualify as '*noncommutative topology*' proper and its related noncommutative sheaf or scheme theory. At the same time, there is no collective agreement on how such a noncommutative or quantum [34, 35, 36, 30, 29] topology may be applied to the problem of the quantum structure and dynamics of spacetime.

[4]The (a) and (b) pathological features of the manifold model above may be summarized in its character as *a geometric point set differential continuum of events* [48].

Furthermore, the (algebras of) commutative $\mathcal{C}^\infty$-determinations of the manifold's point events indicate another non-quantal (classical) feature of the spacetime manifold [61, 48].

- (d) **Additional structures:** M's need of extra structures required to be introduced by hand by the theoretician and not being 'naturally' related to the topological manifold (*ie*, the $\mathcal{C}^0$-continuous) one. Such structures are the differential (*ie*, the $\mathcal{C}^\infty$-smooth) and Lorentzian metric (*ie*, the smooth metric field $g_{\mu\nu}$ of absolute signature 2) ones [5], and they are implicitly postulated by the general relativist on top of M's fixed continuous topology in order to support the apparently necessary full differential geometric (*ie*, Calculus based!) panoply of general relativity. The 4-dimensional, $\mathcal{C}^\infty$-smooth Lorentzian manifold assumption for spacetime concisely summarizes the kinematics of general relativity [73, 47, 62, 48].

- (e) **Non-operationality:** M's gravely non-operational (*ie*, non-algebraic) character as a static, pre-existent background geometric structure—an inert stage on which fields propagate and interact—whose existence is postulated up-front by the theorist rather than being defined by some (algebraically modelled) physical operations of determination or localization of its point events. This seems to be in striking discord with the main tenet of the philosophy of quantum theory supporting an observer (and observation!) dependent reality [57, 47]. Furthermore, one would ultimately expect that *it is the dynamical relations between quanta that define spacetime*, that is to say, from which spacetime, with its topological, differential and Lorentzian metric properties, should be effectively derived somehow [62], so that the latter should not be regarded as an *a priori* absolute ether-like substance [22]—an unjustifiably necessary passive receptacle fixed once and forever to host dynamical fields and their interactions, but, at the same time, an entity that does not actively participate in them[5]. In any case, and in view of (b) above, we have no actual experience of a continuous infinity of events and their differential separation cannot be recorded in the laboratory; for evidently, realistic experiments are of finite duration and are carried out in laboratories of finite size. Moreover, as a matter of principle, one cannot determine the gravitational field, hence the metric separation, between infinitesimally separated events (*ie*, events whose space-time distance is smaller than Planck's— $l_p \approx 10^{-35}m$-$t_P \approx 10^{-44}s$) without creating a black hole. This seems to point to a fundamental cut-off of continuous spacetime which strongly suggests that spacetime becomes reticular or granular above a certain Planck energy ($E_P \approx 10^{-19}GeV$). The continuous commutatively coordinatized geometric manifold is experimentally (or experientially!) a non-pragmatic model of spacetime that should be replaced at a basic level by a physically more plausible, perhaps combinatorial and quantal (*ie*, noncommutative-algebraic), structure [61, 56, 47, 62, 48, 58].

- (f) **Spatiality and globalness of topology:** M's 2-way undirected, locally Euclidean topological structure, will likely prove to be inadequate for modelling the irreversible small scale connections between events, for it has been seriously proposed that the 'real' quantum theory of gravity will turn out to be 'innately' a time-asymmetric theory [54, 27, 32, 28, 58]. At the same time, the very conception of topology as a theory of reversible, spatial (or spacelike!) connections between points should be challenged, and justly so because of the prominent lack of experimental evidence for tachyons moving back and forth in spatial or spacelike directions. In any case, the general conception of topology as the study of the 'global' features of space may seem to be problematic in a fundamental theoresis of Physis where all significant dynamical variables are expected to respect some kind of locality principle (*ie*, where all observables are in effect local variables propagating in temporal or causal directions independently of whether this dynamics ultimately turns out to be time-asymmetric or not).

With these doubts about the physical soundness of the geometric spacetime continuum in the quantum deep, we are able to discuss next a finitary-algebraic model for (the kinematics of) discrete Lorentzian quantum gravity that we presently possess based on sheaf and topos-theoretic ideas, as it were, to alleviate or even evade the aforementioned (a)-(f) 'pathologies' of the classical spacetime manifold.

3 The Present: sheaves and their topoi in discrete Lorentzian quantum gravity

Pointlessness and Discreteness: Finitary substitutes for continuous spacetime topology, that is to say, when spacetime is modelled after a topological (*ie*, $\mathcal{C}^0$) manifold M, were derived in [72] from locally finite

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[5]In the words of Einstein: "*a substance that acts, but is not acted upon*" [22, 23].

open covers of a bounded region X of M in a spirit akin to the combinatorial Čech-Alexandrov simplicial skeletonizations of continuous manifolds [1, 2, 21, 20]. These substitutes were seen to be locally finite T_0-posets and were interpreted as finitary approximations of the continuous locally Euclidean topology of M. At the heart of this approach to spacetime discretization lies the realistic or 'pragmatic' assumption [61, 62] that at a fundamental level the singular spacetime point events should be replaced (or smeared out) by something coarser or 'larger', with immediately obvious candidates being open sets (or generally, 'regions') about them [72, 73, 13]. Then, the resulting poset topological spaces that substitute M are, in fact, complete distributive lattices otherwise known as locales [42]. The pointlessness of these *finitary locales* (finlocales) should be contrasted against the pointedness of M mentioned in (a) above, while their discreteness comes to relieve M's pathology (b). Also, it should be emphasized that the continuous M, with the $\mathcal{C}^0$-manifold topology carried by its points, can be recovered at the ideal inverse limit of infinite refinement of an inverse system or net of these finlocales, so that one is able to establish a connection between these discrete topological poset substrata and the continuous manifold that they replace on the one hand, as well as to justify their qualification as sound approximations of M on the other [72, 56, 47].

Algebra over geometry: In [57], sheaves of continuous functions on the finitary locales of the previous paragraph were studied. In the same way that the locally finite locales were interpreted as sound approximations of the continuous topology of (the bounded region X of) the spacetime manifold M, so their corresponding *finitary spacetime sheaves*[6] were viewed as reticular substitutes of the sheaf of (algebras of) $\mathcal{C}^0$-functions on X which, in turn, represent the *observables of the continuous topology of the spacetime region X*. The duality of the two approaches in [72] and [56] towards discretizing the spacetime topological continuum was particularly emphasized in [56], namely that, while in the former scheme finitary locales approximate the spacetime topology *per se*, in a more *operational* (*ie*, *algebraic*) spirit that suits sheaf theory [43, 44], the latter approach discretizes (or 'finitizes') the sheaf structure of (the algebras of) our own observations of that continuous spacetime topology. The crucial change of emphasis in the physical semantics of the two approaches is from discretizations of a background point set geometric realm 'out there', to ones of our very own (algebraic) operations of perceiving that realm—which spacetime realm, for all we know, may not physically exist independently of these operations after all. Arguably, the finsheaf-theoretic approach comes to alleviate the shortcoming (e) of M above; while sheaves, being by definition *local homeomorphisms* [42, 43, 44, 47], certainly address M's 'globalness' problem alluded to in (f), namely, *local topological information is more important than the usual global information* about, say, handles, holes *etc* that (the 'classical' conception of) topology is concerned with [28, 47, 62].

Temporality and causality over spatiality and topology: In a dramatic change of physical interpretation of the finitary topological posets (finlocales) involved in [72], Sorkin and coworkers insisted that the partial orders involved should not be interpreted as coarse topological or undirected 2-way spatial relations between geometric points, but rather as directed 1-way primordial causal 'after' relations between events inhabiting so-called *causal set* substrata[7] that are supposed to fundamentally underlie the classical curved Lorentzian manifold of general relativity at Planck scales [5, 70, 71, 73, 74, 64, 75]. In contradistinction to the purely topological character of the finitary locales in [72], causets are supposed to encode information about the microcausal relations between events in the quantum spacetime deep. Moreover, as Sorkin *et al.* stress in [5], the partial order causality relation of causets encodes almost complete information not only about the topological (*ie*, $\mathcal{C}^0$) structure of the classical spacetime manifold, but also about its differential (*ie*, the $\mathcal{C}^\infty$-smooth) and conformal Lorentzian metric structures (*ie*, the metric $g_{\mu\nu}$ of signature 2 modulo its determinant which represents the elementary spacetime volume measure) that are usually externally prescribed by the theorist on top of its continuous topology. That causality in its order-theoretic guise is a deeper, more physical[8] (and perhaps more pertinent to the problem of quantum gravity) conception than topology *per se* has already been amply noted in [83, 84, 27, 29, 74, 55, 47, 75]. The upshot of the aforementioned 'semantic reversal' is that from the causet viewpoint, locally finite partial orders should not be viewed as effective topological approximations of the classical spacetime manifold, but, on the contrary, the latter should be regarded as being of a contingent (*ie*, non-fundamental) character, and as reflecting our own ignorance about (and related 'grossness' of our model of) the very fine structure of the world. All in all, the manifold is the poor relative, ultimately, the approximation of the causet, not the other way around.

To recapitulate then, partial orders as causal, not topological, relations: this is what is 'going on' between events in the quantum deep [75]. All this certainly presents a sound alternative to the 'additional structures' and 'spatiality' problems of the spacetime manifold mentioned above in (d) and (f), respectively.

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[6] We will call them 'finsheaves' for short *à la* [47].

[7] Hereafter to be abbreviated as 'causets' [47].

[8] Especially due to lack of experimental evidence for tachyons. Again, see section 2.

Finlocales and their corresponding causets quantized: In [61], an algebraic representation of Sorkin's finlocales was given, namely, with every finitary poset substitute of a continuous spacetime manifold a finite dimensional, complex, associative and noncommutative *Rota incidence algebra* [66, 76, 53] was associated in such a way that the topological information encoded in the former was seen to be the same as that encoded in the latter [9]. Furthermore, in the new environment of the Rota algebras there is a natural linear superposition operation between the arrows (*ie*, the partial order relations) in their corresponding posets that is characteristically absent from Sorkin's formulation of discrete topological spaces as posets [72]. In other words, and this is the main physical interpretation of the formal mathematical structures involved in [61], in the algebraic context one is able to form *coherent quantum superpositions between the topological connections defining these reticular topological substrata of the classical spacetime manifold*. Moreover, this interpretation of the incidence algebras associated with the finlocale replacements of the classical continuum as *discrete quantum spacetime topologies* enabled us to conceive of the aforementioned inverse limit procedure by which the continuum is recovered from finlocales in [72] as *Bohr's correspondence principle*. That is, the topological spacetime manifold arises at the classical and experientially non-pragmatic limit of infinite energy of resolution [15], and concomitant 'decoherence', of an inverse system of reticular quantum topological Rota algebraic substrata [61, 62]. Thus, in the Rota finitary-algebraic context we are able to formulate a quantum sort of spacetime topology [58] hence evade the problematic non-quantal nature of the continuum mentioned in (c) above.

In connection with this continuum classical limit, it should also be mentioned that actually not only the $\mathcal{C}^0$-topological, but also the differential (*ie*, the $\mathcal{C}^\infty$-smooth) structure of spacetime was anticipated in [61, 62] to emerge at the classical limit from a 'foam' of such discrete quantum Rota topologies. This is so because the incidence algebras under focus in [61, 62] were seen to be *graded discrete differential manifolds* in the sense of Dimakis and Müller-Hoissen [19, 18, 3, 10][9]. Moreover, since the differential structure of the limit manifold represents the notion of *locality* in classical spacetime physics[10] [22], these algebraic discrete quantum topological substrata were coined '*alocal structures*'—as mentioned earlier, in a sense neither local (general relativity) nor non-local (quantum mechanics) structures [61, 62].

A couple more things should be mentioned now that we are talking about the Rota algebraic quantization of Sorkin's finitary locales. First, one should emphasize that the general method (and philosophy!), originally due to Gelfand, of extracting points from algebras as well as of assigning a fairly 'natural' topology to the latter, thus 'geometrizing', as it were, algebraic structures, was first used by Zapatrin in [81] for gathering useful geometrical information from finite dimensional incidence algebras and for establishing their topological equivalence to the finitary posets of Sorkin [9]. At the heart of this so-called 'spatialization procedure' lies the recognition that points in these algebras are precisely the (kernels of equivalence classes of) irreducible (finite dimensional Hilbert space) representations of these algebras which, in turn, may be identified with the elements of their primitive spectra (*ie*, the primitive ideals in the algebras) [61, 62, 58].

The second thing that should be noted here is the categorical duality (*ie*, a contravariant functor) between the poset category of incidence Rota algebras associated with the finitary locales of Sorkin, and the poset category of the latter when viewed as simplicial complexes *à la* Čech-Alexandrov[11] [61, 82, 62, 48, 58]. In [58] the latter category, consisting of finitary posets or simplicial complexes and 'refinement arrows' $\preceq$[12], was called the Alexandrov-Sorkin category[13], while the former category, consisting of finite dimensional incidence algebras and 'coarsening arrows' $\succeq$[14], was coined the Rota-Zapatrin category[15]. For the time being we note that this contravariant functor between $\mathfrak{P}$ and $\mathfrak{R}$ may be immediately recognized as defining *a presheaf of finite dimensional incidence algebras over finitary locales*.

We should also mention that in [56] an algebraic quantization procedure of the locally finite poset structures representing causets of Sorkin *et al.* [5] was suggested based on the analogous process of quantization of finlocales of Sorkin [72] proposed in [61]. In little detail, with every causet its incidence Rota algebra was associated and interpreted as a *quantum causal set*[16] in such a way that the local causal-topological information encoded in the causet[17] corresponds to the one encoded in the generating relations of the algebraic

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[9]In a nutshell, a reticular analogue of the nilpotent Kähler-Cartan differential d (and its dual homological boundary operator δ) can be defined on these incidence algebras. See [82, 48].

[10]That is, the local structure of classical spacetime is taken to be the point event and the space (graded module) of differential forms (co)tangent to it.

[11]See [1, 2, 21, 20] for this so-called 'nerve construction of simplicial complexes'.

[12]That is, injective simplicial maps or injective poset morphisms, or even, 'continuous injections' between finitary locales.

[13]Symbolized by $\mathfrak{P}$.

[14]That is epi incidence algebra homomorphisms.

[15]Symbolized by $\mathfrak{R}$.

[16]Hereafter to be referred to as '*qauset*'.

[17]That is, the info in the so-called 'covering relations'—the immediate causal arrows of the underlying poset Hasse graphs.

Rota topology of the qauset[18]. Another important thing to notice from [56] in the Rota algebraic environment that we have cast causets, and this is the main virtue of qausets that essentially qualifies them as the quantum analogues of causets, is that the model allows for coherent quantum superpositions between the causal arrows—a feature that was prominently absent from the purely poset categorical (arrow semigroup) structures modelling causets. Thus, we have in our hands a finitary-algebraic model for quantum causal topology [58].

Curving a noncommutative topology for qausality: In [47], curved finsheaves of qausets were defined as principal finsheaves of the non-abelian incidence Rota algebras modelling qausets having for structure group of local symmetries a finitary version of the continuous orthochronous Lorentz group and for base or localization space Sorkin *et al.*'s causets[19]. Non-trivial spin-Lorentzian (*ie*, $sl(2,\mathbb{C})$-valued) connections on these finsheaves were defined *à la* Mallios [43, 44, 48], and the resulting structures were interpreted as finitary, causal and quantal substitutes of the kinematics of Lorentzian gravity since an inverse system of these finsheaves was seen to 'converge' in the limit of infinite energy of localization to the Lorentzian manifold—the kinematical structure of general relativity [47].

Then, it has been recently speculated [47, 59, 48, 58] that as $\mathbf{Sh}(X)$—the topos of sheaves of sets over a spacetime manifold X—may be viewed as a (mathematical) universe of variable sets varying continuously over X [42], so a possible topos-organization of the curved finsheaves of qausets in [47] (call it $^{fcq}\mathbf{Sh}(X)(\vec{P})$[20]) may be regarded as a (physical) universe of dynamically variable qausets varying under the influence of a locally finite, causal and quantal version of Lorentzian gravity. In such a possible model it would be rather natural to address the question opening the present paper since the internal intuitionistic-type of logic of $^{fcq}\mathbf{Sh}(X)(\vec{P})$ should be intimately related to the intuitionistic logic that underlies quantum logic proper in its topos-theoretic guise [11, 12, 14]. This gives us significant hints for the deep connection between the quantum logical structure of the world and its dynamically variable reticular causal or chronological structure at Planck scales *vis-à-vis* quantum gravity.

The discussion above brings us to the use of presheaves and their topoi in quantum logic proper and, *in extenso*, to the logic of consistent-histories.

Presheaves and their topoi in quantum logic and consistent-histories: In [11, 12, 14], the Kochen-Specker theorem of quantum logic was studied from a topos-theoretic perspective. In particular, it was shown that quantum logic is 'warped' or 'curved' relative to its Boolean sublogics [63]. This was achieved by showing that certain presheaves of sets over the base poset category of Boolean subalgebras of a quantum projection lattice $\mathcal{L}$ associated with the Hilbert space $\mathcal{H}$ (of dimensionality greater than 2) of a quantum system do not admit global sections, but they do so only locally. Since these sections were interpreted as valuations (on propositions represented by the projectors in $\mathcal{L}(\mathcal{H})$), and since the presheaves were organized into a topos (of so-called 'varying sets' [40]), the aforesaid warping phenomenon could be read as follows: unlike classical (Boolean) logic—which is the internal logic of the 'classical' topos **Set** of constant sets, quantum logic does not admit a global notion of truth; or equivalently: in quantum logic truth is localized on (or relativized with respect to) the Boolean logics embedded in it. Furthermore, as a result of this, and as befits the internal logic of the topos of presheaves of sets over a poset category [31, 39, 67, 68, 42], Butterfield *et al.* show that quantum logic is locally intuitionistic ('neorealist'), not Boolean ('realist').

Very similar to the treatment of quantum logic by presheaf and topos-theoretic means above is Isham's assumption of a topos-theoretic perspective on the logic of the consistent-histories approach to quantum theory [38]. Briefly, Isham showed that the universal orthoalgebra $\mathcal{U}\mathcal{P}$ of history propositions admits non-trivial localizations or 'contextualizations' (of truth) over its classical Boolean subalgebras. More technically speaking, it was shown that one cannot meaningfully assign truth or semantic values to propositions about histories globally in $\mathcal{U}\mathcal{P}$, but that one can only do so locally, that is to say, when the propositions live in certain Boolean sublattices of $\mathcal{U}\mathcal{P}$—the classical sites, or 'windows' [11, 12, 14], or even 'points' [50, 51, 58] within the ortholattice $\mathcal{U}\mathcal{P}$. Moreover, the simultaneous consideration of *all* such Boolean subalgebras and *all* consistent sets of history propositions led Isham to realize that the internal logic of the consistent-histories theory is neither classical (Boolean) nor quantum proper, but intuitionistic[21]. This result befits the fact that the relevant mathematical structure involved in [38], namely, the collection of presheaves of

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[18]This observation will be of crucial importance in the next paragraph where we will talk about finsheaves of qausets as local homeomorphisms between causets and qausets [47].

[19]Technically speaking, finsheaves of qausets over causets are local homeomorphisms between the base causets and the qauset stalks [47]. The aforementioned local topological equivalence between finlocales and their incidence algebras comes in handy for defining such finitary local homeomorphisms (finsheaves).

[20]The topos of sheaves of (f)initary, (c)ausal and (q)uantal sets (qausets) over Sorkin *et al.*'s causets $\vec{P}$ [47].

[21]As alluded to above in the context of quantum logic proper, Isham in [38] uses the epithet 'neorealist' for the quantal logic of the consistent-histories theory in its topos-theoretic guise. Quite resonably, we feel, one could also coin this logic 'neoclassical' [58]—this name referring to the departure of the Brouwerian logic of the topos of consistent-histories in [38] from

sets varying over the poset category of Boolean sublattices of $\mathcal{UP}$, is an example of a topos [40, 4, 42], for it is a general result in category theory that every topos has an internal logic that is strongly typed and intuitionistic [31, 39, 67, 68, 42]. As in the case of quantum logic, the logic of consistent-histories is (locally) intuitionistic (Heyting) and 'warped' relative to its 'local' classical Boolean sublogics [63][22].

Furthermore, in [59], the base poset category of Boolean sublattices of $\mathcal{UP}$ was endowed with a suitable Vietoris-type of topology so that the presheaves of varying sets over the Boolean subalgebras of $\mathcal{UP}$ were appropriately converted to sheaves and, as a result, their respective topos was viewed as a mathematical universe of sets varying *continuously* over $\mathcal{UP}$. Moreover, the stalks of these sheaves were given further algebraic structure—that of incidence Rota algebras—together with the latter's physical interpretation as qausets [56, 47], so that we arrived at *sheaves of consistent-histories of qausets*. As a result, the topos-like organization of these sheaves—the *topos of quantum causal histories*—was anticipated to be the natural physico-mathematical universe in which 'curved quantum causality meets warped quantum logic'[23], as it were, to answer to the first question opening the present paper.

Now, the discussion about the topos of quantum causal histories brings us to speculate briefly about the immediate future development of our general research project 'presheaves, sheaves and their topoi in quantum gravity and quantum logic'.

4   The Future: envisaging 'quantum sheaves' and their 'quantum topoi'

Of great interest to us for the future development of our research program, and keeping in mind the second question opening this paper, is the following project: since the incidence algebras modelling qausets are graded non-abelian Polynomial Identity (PI) rings, it would in principle be possible to develop a noncommutative sheaf or scheme type of theory [33, 24, 69] for such finitary non-abelian PI ring localizations. Rigorous mathematical results, cast in a general categorical setting, from the noncommutative algebraic geometry of similar non-abelian schematic algebras and their localizations [79, 77] are expected to deepen our physical understanding of the dynamically variable noncommutative quantum causal Rota topologies defined on the primitive spectra of qausets [61, 47, 62, 58][24]. Ultimately, the deep connection for physics is anticipated to be one between such a noncommutative conception of the causal topology of spacetime and the fundamental quantum time-asymmetry expected of the "*true quantum gravity*" [54, 27, 32]. The deep connection for mathematics is, as we briefly mentioned in section 2, that such a general conception of a 'noncommutative topology' is supposed to be the precursor to Connes' 'noncommutative geometry'[25] [16]—a theory that in the last five years or so has become of great interest to theoretical physics, because it appears to shed more light on the persisting problem of quantum gravity. For we emphasize again: it seems unreasonable to have a full fledged noncommutative geometry and lack a noncommutative topology and its corresponding sheaf theory, especially to apply the latter to quantum gravity where even the spacetime topology is expected to be subject to quantum dynamical fluctuations and coherent superpositions [62].

This 'noncommutative quantum causal topology' project has revived this author's doctoral interests and work in topoi, their possible quantization and the application of the resulting 'quantum topoi' to the problem of the quantum structure and dynamics of spacetime [55]. In particular, but briefly, Finkelstein [29], as part of an ongoing effort to find a quantum replacement for the spacetime manifold of macroscopic physics, has developed a theory of quantum sets, which in a sense represents a quantization of ordinary 'classical' set theory. The basic idea is that spacetime at small scales should really be viewed as a 'quantum' set, not a classical one. This is supposed to be a step on the path to a 'correct' version of quantum gravity and quantum spacetime topology [30]. A question which may occur to a modern logician or 'toposopher' is: what is so special about the category **Set** of classical constant sets, since there are other logical universes just as good, and possibly better, namely 'topoi'? Perhaps it would be a better idea to try and quantize these more general categories, since the use of **Set** may be prey to classical chauvinism.

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the two-valued Boolean lattice calculus obeyed by the states of a classical mechanical system which are modelled after point subsets of its phase space. Arguably then, the logic of a classical mechanical system is Boolean like that of the topos **Set** of constant 'classical' sets.

[22]This departure of quantum logic proper [11, 12, 14] and of the quantal logic underlying consistent-histories [38] from classical Boolean logic is certainly less striking than the famous 'global' difference between quantum and Boolean logic, namely that, while the latter is distributive, the former are non-distributive. Thus, properly speaking, quantum logic, although it is globally non-distributive, locally it is so; albeit, non-Boolean, but intuitionistic (Heyting).

[23]See title of the talk delivered at QS5 (read first footnote in this paper).

[24]I wish to thank Professor Fred Van Oystaeyen (Antwerp University, Belgium) for motivating such a study in a crucial and timely private communication, and in two research seminars—see [78].

[25]Freddy Van Oystaeyen in private correspondence.

The usual flat (*ie*, in the absence of gravity) classical and quantum field theories are conveniently formulated in **Set**, or more precisely, in $\mathbf{Sh}(X)$-the 'classical' topos of sheaves of sets varying continuously over the classical spacetime manifold X [67, 68]. However, as we said in section 2, these theories suffer from non-renormalizable infinities coming from singularities that plague the smooth spacetime continuum. The manifold model, as an inert classical pointed geometric background continuum on which fields propagate and interact, must at least be revised in view of the pathological nature of quantum gravity when treated as another quantum field theory[26] [47]. Topoi and their topological relatives, locales, which are pointless topological spaces, are structures well-suited not to significantly commit themselves to the pathological geometric point-like character of a base spacetime manifold. As it has already been pointed out, perhaps one could arrive at the 'true' topos of Nature, on which a finite quantum theory of gravity can be founded, by considering the pointless topos of the curved finsheaves of qausets over Sorkin's causets, or even the topos of sheaves of quantum causal histories, instead of their classical ancestor $\mathbf{Sh}(X)$. This quest for the 'right' quantum topos of Nature is also expected to shed more light on the following analogy that has puzzled mathematicians for quite some time now:

$$\frac{\text{locales}}{\text{quantales}} = \frac{\text{topoi}}{?}[27]$$

To dwell briefly on this analogy, topologically speaking any complete distributive lattice is called a locale[28] and it corresponds to a generalized (*ie*, 'pointless') topological space [42]. A quantale [6, 7, 8, 65, 49, 52, 50, 51], the noncommutative (quantum) analogue of a locale, may be represented by the lattice of closed two-sided ideals of a nonabelian (von Neumann or C^*) algebra. The primitive spectra of non-abelian qausets [61, 56, 62], when regarded as some sort of lattices[29], may also be viewed as some kind of quantales—albeit, of a finitary sort [58], hence our regarding the topos of finsheaves of qausets (or the sheaves of quantum causal histories) as a strong candidate for the elusive quantum topos.

In the same line of thought, and in connection with sheaves of qausets over consistent-histories and their possible topos-organization mentioned at the end of the previous section, we would like to mention another project that we are currently working on[30]. One may recall that in Isham's version of the quantal logic of consistent-histories [37, 38] central role is played by the tensor product '$\otimes$' structure. A sheaf $\mathfrak{H}$ of Hilbert spaces $\mathcal{H}$ over a classical spacetime manifold X was initially expected to be the appropriate mathematical structure to model Isham's scenario. However, the tensor product $\otimes$ and the 'classical' definition of a sheaf (of tensor product $\mathcal{H}$-spaces[31]) do not seem to go hand in hand for the following, at least from a physical point of view, reason: when one considers the tensor product of two distinct stalks in a vector sheaf like $\mathfrak{H}$, as when one combines two distinct quanta in the usual quantum theory[32], the two stalks 'collapse' to a tensor product stalk over a single spacetime point event of the classical base spacetime manifold X. This phenomenon is characteristic in both classical and quantum field theories (in the absence of gravity) where, when we combine or entangle systems by tensor multiplication, their spacetime coordinates combine by identification. "*This mathematical practice expresses a certain physical practice: to learn the time, we do not look at the system but at the sun (or nowdays) at the laboratory clock, both prominent parts of the episystem*" [29], and it should be emphasized that the episystem is always regarded as being classical[33] in the sense of Bohr. Thus, we expect that the formulation of some sort of '*quantum sheaf*' is required in order to be able to model non-trivially quantum entanglement; moreover, it is quite reasonable to assume that such a quantum notion of a sheaf will be accompanied by an appropriate quantum notion of spacetime topology on which such sheaves are soldered.

We conclude this paper by mentioning another potential application of sheaf and topos theory to quantum gravity that only lately we have envisaged and started to comprehend in full [48]. It concerns the possible application of sheaf and topos theory towards formulating an abstract sort of differential geometry *à la* Mallios [43, 44, 45, 46] on the aforementioned curved finsheaves of qausets or their related sheaves of quantum causal histories, as it were, to transcribe most of the differential geometric apparatus of $\mathcal{C}^\infty$-smooth manifolds[34] to a reticular-algebraic setting that is *ab initio* free from the former's pathological

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[26]That is, 'Quantum Gravity as Quantum General Relativity'.

[28]Logically speaking, a complete Heyting algebra [31, 39, 4, 42].

[29]That is, if on top of their partial order structure, $\cap$ and $\cup$-like operations are defined in them as in the case of the finitary topological spaces (finlocales) mentioned in the previous section.

[30]In collaboration with Chris Isham.

[31]Such sheaves may be coined 'Fock sheaves' for obvious reasons.

[32]In the sheaf $\mathfrak{H}(X)$, local states of quanta are supposed to be represented by its local sections.

[33]Here, the classical base spacetime manifold X.

[34]Arguably, the mathematical apparatus on which general relativity relies.

infinities and incurable diseases. For instance, we have been able to perform a finitary version of the usual $\mathcal{C}^{\infty}$-smooth Čech-de Rham cohomology, as well as initiate a finsheaf-cohomological classification of the non-trivial finitary spin-Lorentzian connections dwelling on the curved finsheaves of qausets in [47]. In this context, what we would also like to work on in the immediate future is to try to relate Mallios' Abstract Differential Geometry [43, 44] and its finitary applications in [48] with the Kock-Lawvere Synthetic Differential Geometry [41] and its promising topos-theoretic applications to quantum gravity [13]. In this respect however, the quest has just begun.

Acknowledgments

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